NOTE ON THE DEFINITION OF AN ABELIAN GROUP BY INDEPENDENT POSTULATES

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E. V. HUNTINGTON has given two definitions of an Abelian group by sets of three and four independent postulates respectively.* I propose two definitions by sets of two and three independent postulates. These definitions are explicitly modelled after those of Huntington; they differ from the latter, however, in that the associative and commutative laws are replaced by a single law, logically as simple as either, and furnishing (in connection with the other postulates of each set) a necessary and sufficient condition for the subsistence of both the associative and commutative laws.

For convenience of reference, I shall re-state Huntington's first definition :—

A system made up of a set of elements, a, b, c, $\cdots$, with a rule of combination, $\circ$, is called an Abelian group when the following conditions are satisfied :†

$H1$). $a \circ b = b \circ a$, *whenever* a, b *and* $b \circ a$ *belong to the set.*

$H2$). $(a \circ b) \circ c = a \circ (b \circ c)$, *whenever* a, b, c, $a \circ b$, $b \circ c$ *and* $a \circ (b \circ c)$ *belong to the set.*

$H3$). *For every two elements* a *and* b *there is an element* x *in the set such that* $a \circ x = b$.

The distinction between finite Abelian groups of various orders and infinite Abelian groups is made by adding :—

a). *The set contains* n *elements;* or

b). *The set is infinite.*

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* *Transactions of the American Mathematical Society*, vol. 4 (1903), pp. 27-30. For bibliographical references to other definitions, see *ibid*, vol. 6 (1905), p. 181.

† While the usage in regard to the words "set" and "system" is not wholly uniform, it seems desirable to use "set" or "class" for any collection of objects regarded simply as distinguishable entities, and "system" for a collection of objects *plus* some relations or rules of combination.

1. Definition by two postulates. I define an Abelian group by the following postulates :—

1). $(a \circ b) \circ c = a \circ (c \circ b)$, *whenever* $a, b, c, a \circ b, c \circ b$ *and* $(a \circ b) \circ c$ *belong to the set.*

2). *For every two elements* a *and* b *there is an element* x *in the set such that* $a \circ x = b$.

Postulate 2) is precisely $H3$). Thus, to justify the characterization of the sytem just defined as an Abelian group, it will suffice to deduce $H1$), $H2$) from 1), 2).

Proof of $H1$). Suppose a, b and $b \circ a$ belong to the set. By 2), take x so that $a \circ x = b$. Then by 1), since a, x, $a \circ x$, and $(a \circ x) \circ a$ belong to the set,

$$(a \circ x) \circ a = a \circ (a \circ x) ;$$

that is,

$$b \circ a = a \circ b.$$

Proof of $H2$). Suppose a, b, c, $a \circ b$, $b \circ c$, and $a \circ (b \circ c)$ belong to the set. Then by $H1$),

$$a \circ (b \circ c) = (b \circ c) \circ a = (c \circ b) \circ a,$$

and by 1),

$$(c \circ b) \circ a = c \circ (a \circ b).$$

Again, by $H1$),

$$c \circ (a \circ b) = (a \circ b) \circ c ;$$

so that

$$a \circ (b \circ c) = (a \circ b) \circ c.$$

Therefore an Abelian group is defined by 1), 2).

Independence of 1), 2), *and* a).

The postulates 1), 2), a) are independent when $n > 1$. The proof is established in the usual way by the following systems :

(1). The first n positive integers, with $a \circ b = b$; or the n distinct integers modulo n, with $a \circ b \equiv -a - b \pmod{n}$. The former example is associative but not commutative, the latter is commutative but not associative.

(2). The first n positive integers, with $a \circ b = 1$.

(a). Any infinite Abelian group.

Independence of 1), 2), *and* b).

Consider in the same manner, the following systems:

(1). All positive integers, with $a \circ b = b$; or again, all rationals, with $a \circ b = 2(a + b)$.

(2). All positive integers, with $a \circ b = a + b$.

(b). Any finite Abelian group.

2. Definition by three postulates. The set of postulates modelled after Huntington's second set is as follows:

1′). $(a \circ b) \circ c = a \circ (c \circ b)$, *whenever* a, b, c, $a \circ b$, $c \circ b$, $(a \circ b) \circ c$, *and* $a \circ (c \circ b)$ *belong to the set.*

2′). *For every two elements* a *and* b, *there is an element* x' *in the set such that* $(a \circ x') \circ b = b$.

3′). *If* a *and* b *belong to the set, then* $a \circ b$ *belongs to the set.*

That 1′), 2′), 3′) really define an Abelian group is made clear by noting that from them may be deduced 2), by taking $x = b \circ x'$.

Independence of 1′), 2′), 3′), *and* a).

The non-group systems (1′), (2′), (a) are the same as (1), (2), (a), used previously.

For (3′), take the first n positive integers, with $a \circ b = 1$, unless $a = 1$ or $b = 1$; $1 \circ b = b$; $a \circ 1$ not in the set unless $a = 1$. Here 2′) is satisfied when $x' = a$.

Independence of 1′), 2′), 3′), *and* b).

The non-group systems (1′), (2′), (b) are the same as (1), (2), (b), used previously.

For (3′), take all integers with $a \circ b = a + b$ when a or b or $a + b$ is zero; otherwise $a \circ b$ not in the set.

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